Model of compactons on jet streams and their collapse

V. Goncharov*
Institute of Atmospheric Physics, Russian Academy of Sciences, 109017 Moscow, Russia

V. Pavlov†
UFR de Mathématiques Pures et Appliquées-LML UMR 8107, Université de Lille 1, 59655 Villeneuve d'Ascq, France
(Received 30 November 2006; revised manuscript received 26 October 2007; published 20 December 2007)

We formulate the Hamiltonian version of contour dynamics for a model of axially symmetric, equally vortexed jet streams with a free boundary. In particular, we study dominant structural elements which appear in strongly perturbed jet streams at the stage of their decay. The model produces solutions with compact support which can describe such dominant structures, called compactons. Evolution of the compactons can lead to their collapse which virtually does not deform the shape, but gradually intensifies the vortex sheet at the boundary according to the law $(t_0-t)^{-1}$, where t_0 is the collapse time.

DOI: 10.1103/PhysRevE.76.066314 PACS number(s): 47.32.C−, 47.10.Df, 05.45.Yv

I. INTRODUCTION

Physics of high density energies, such as astro -or geophysics, often employ models studying behavior of the moving boundaries where fluid density experiences a noncontinuous jump. Indeed, in the middle of the past century E. Fermi and J. von Neumann extensively studied instability at the boundary of two fluids in connection with the *im*plosive method to nuclear blast. The problem involved instability arising when heavy fluid is decelerated by the lighter one. As Fermi noted, this problem is physically equivalent to the one where heavy medium is located above the light one in the uniform gravity field [1]. This instability was explored in the framework of the simplest model where only the density jump occurred [2–6].

One can envision more complex models which study the dynamics of instability of the boundaries with jumps not only of media density, but also of other field variables such as vorticity. The existence of such boundaries leads to various forms of hydrodynamic instabilities [7] which in the nonlinear stages of their development cause disruption of fluid layers and formation of localized spatial structures. These structures emerge on the backdrop of turbulent fluctuations as regions of coherent large-scale flow of the fluid with high concentration of vorticity.

These striking phenomena can be observed in the nature around us. Occasionally, stratified flows with very large Reynolds number (for example, in atmospheres and oceans) spontaneously organize into large scale vortex structures such as thermics, jets, bubbles, etc. However, the traditional description of hydrodynamic mixing in stratified media in terms of diffusion is imprecise and does not provide information about such a substantial feature of layered medium mixing as the very existence of these large dominant structures.

One of the approaches to studying the stratified fluid dynamics is to employ idealized models where the surfaces of constant properties move with the fluid. Then the evolution of the system can be described in terms of variables defining only the boundary, while ignoring the description of the rest of the fluid. This approach is known as the *contour dynamics method* (CDM) [8–18]. A classical example of its application is the model of surface waves on an incompressible fluid.

In the studies where the hydrodynamic system is weakly perturbed and the governing equations can be solved using the perturbation theory, the spectral mode concept may describe the development of instability in principle, but only at the initial stage of the evolution process. However, when the instability develops and the magnitude of field perturbations becomes comparable with the characteristic value of the basic state, the traditional spectral mode concept becomes inapplicable and requires transitioning from the analysis of the evolution of separate harmonics to the analysis of dominant structures which describe the general picture of the nonlinear mixing process prior to the beginning of turbulence. Thus different approaches should be employed beyond weak nonlinearity.

For the above-mentioned reasons, development of new analytical methods and construction of adequate models describing such large-scale turbulent structures, are crucial for understanding of mixing processes. To date, many numerical and analytical studies of the dominant structures have been performed (see Ref. [7] and references therein). However, since precise study of such structures requires laborious solving of complex systems of nonlinear equations, it is often more effective to use simplified models in order to capture the tendency of the processes [6]. In fact, instead of massaging numerically calculated details of the vortex motion, it is more important to first answer the questions of whether coherent structures may form under specific conditions. And if so, what are the key features of their evolution?

The present article focuses on studying the large-scale vortex structures in the framework which neglects all dissipative processes. Such idealization is justified when, on the one hand, the characteristic spatial scale is greater than the scale of turbulent pulsations, and on the other hand, when direct molecular viscosity can be neglected. Many real-life atmospheric or oceanic flows, including turbulent ones, fit such description. In this framework the true effects of vis-

*v.goncharov@mtu-net.ru
†Vadim.Pavlov@univ-lille1.fr

cosity and turbulence on the vortex structures may be captured via their impact on the evolution of the mean background flow. For example, these effects can lead to formation of a boundary layer, or to flattening of the turbulent velocity profile of the fluid flow in a tube.

The assumption that the vortex structures are large-scale permits a series of crucial simplifications *even* for the fields which continuously distributed in space. First, if limited to two-dimensional flows, we can approximate the medium which has a continuous profile of the field variable such as vorticity, with a stratified model where each layer has constant vorticity. Since large-scale flows are only weakly sensitive to the fine structure of the profile, such an approximation is valid. Consequently, even with a crude model having just a few layers that reflect only the general structure of the real profile, such an approach should reveal the qualitative essence of the large-scale dynamics. Second, stratified models present an opportunity to describe the flow dynamics in the framework of spatially one-dimensional nonlinear integro-differential equations in terms of the boundaries defining the contours of vortex structures (i.e., using the contour dynamics method mentioned above).

To derive the contour dynamics equations, it is advantageous to use the Hamiltonian description which not only provides the means for assessing how universal the approximation method may be, but also helps minimize the technical efforts of solving the problem. Indeed, in this framework the object of all approximation procedures is not the set of many equations, but a single variable—the Hamiltonian. Moreover, in many circumstances it is possible to step beyond the weak nonlinearity in terms of the magnitude of perturbations. The required approximation procedure assumes not the smallness of deviation of dynamic variables from the mean, but the smallness of contributions associated with their spatial derivatives. Therefore, the power series are formed not based on the perturbation magnitudes, but based on the spatial derivatives of the field variables. The lowest order of the perturbation theory preserves in the Hamiltonian only the smallest terms with respect to these derivatives.

Finally, our approach was partially inspired by the remarkable papers of Fermi [3–5] in which he examined surface instability of heavy, incompressible fluids during the *nonlinear* stage of instability development when the amplitude of wave perturbation becomes comparable with the characteristic space scale of the wave. Because, as mentioned above, this problem was important for nuclear *im*plosion, it was studied at one time in great detail. Fermi proposed an elegant method to describe the key features of the process using a very simplified model. This method can be referred to as the method of trial functions. The essence of the approach was to use the principle of the least action for a fluid motion possessing the continual number of degrees of freedom and to parameterize the *field* Lagrangian into the action integral using a *small number* of parameters (generalized coordinates) for which the Lagrangian or Hamiltonian equations are deduced from the extremum of the action.

In the spirit of the idea of Fermi, we attempt to analyze the general picture of large vortex structures evolution. Specifically, we study the dominant structures appearing in axially symmetric, equally vortexed, free-bounded jet streams. We intend to show that the strongly perturbed jet of the equally vortexed fluid with the Attwood number close to 1 (large density contrast at the interface) can be broken into separate vortex blobs called *compactons*. Visually the model can be described as a jet of vortexed fluid which due to the developed instability eventually breaks down into vortex "droplets." Moreover, our numerical simulations (Sec. IV) show that these droplets line up according to their size. The line order starts with the largest compacton and concludes with the smallest (as is with solitons).

In this article, we focus on a rather narrow range of theoretical problems: How to apply the Hamiltonian approach to a system with continual number of degrees of freedom, how to construct the compacton solutions describing the dominant structures, and how to qualitatively describe the mechanism of their stability or instability. In the followup paper we will provide a more complete analysis of the compacton collapse and the numerical simulation of how the dominant structures interact.

This article is organized as follows. In Sec. II we discuss the model setup and formulate the governing nonlinear equations in curvilinear coordinates. In Sec. III we consider the axially symmetric model with free boundary. Numerical calculations showing the evolution of an initial perturbation is given in Sec. IV. The properties of the obtained localized solutions, solitons and compactons, are considered in Sec. V. The problem of compacton instability is considered in Sec. VI. This problem is of particular significance because compacton can collapse, i.e., form a singularity in a finite time. This collapse phenomenon can be manifested as a permanent self-constriction of the localized disturbance which must be accompanied by the infinite increase of its amplitude at the fixed integrals of motion. In Sec. VII we analyze the occurrence of such a collapse due to generation of an axisymmetric mode in the vortex sheet. The self-similar collapse scenario is discussed in Sec. VIII. In Sec. IX we summarize our results. Appendixes A–D list additional relevant data that we hope are useful for the most inquisitive readers.

II. GOVERNING EQUATIONS

A. General consideration

The well-known evolution equations of continuity and momentum for a *perfect incompressible inhomogeneous* fluid are

$$\partial_t\rho + v_k\partial_k\rho = 0, \quad \partial_t\pi_i + \partial_k(v_i\pi_k) = -\partial_i p. \tag{1}$$

Here, density field ρ and momentum field $\boldsymbol{\pi}=\rho\mathbf{v}$ are the dynamical variables of the problem, $\mathbf{v}$ is the flow velocity, div $\mathbf{v}=0$, and p is pressure.

Equation (1) can be written (see Appendix A) in the Hamiltonian form with local functional Poisson brackets $\{\pi_i,\pi_k'\},\{\rho,\pi_k'\}$, and $\{\rho,\rho'\}=0$ as

$$\partial_t\rho = \{\rho,\mathcal{H}\} = \int d\mathbf{x}'\left[\{\rho,\rho'\}\frac{\delta\mathcal{H}}{\delta\rho'} + \{\rho,\pi_j'\}\frac{\delta\mathcal{H}}{\delta\pi_j'}\right],$$

$$\partial_t \pi_i = \{\pi_i, \mathcal{H}\} = \int d\mathbf{x}' \left[\{\pi_i, \rho'\} \frac{\delta \mathcal{H}}{\delta \rho'} + \{\pi_i, \pi_j'\} \frac{\delta \mathcal{H}}{\delta \pi_j'} \right]. \quad (2)$$

Hereinafter, summation over repeated indices is implied and primed field variables represent the dependence on the primed spatial coordinates. Hamiltonian $\mathcal{H}$ in Eq. (2) for the inhomogeneous incompressible fluid is given by

$$\mathcal{H} = \int d\mathbf{x} \frac{\boldsymbol{\pi}^2}{2\rho} = \mathcal{H}[\rho, \boldsymbol{\pi}] \quad (3)$$

and is congruent to kinetic energy of the system expressed in terms of dynamical variables ρ and $\boldsymbol{\pi}$.

In contrast with the traditional methods, the Hamiltonian approach provides not only the ability to adequately select dynamic variables, but also to control the internal symmetries, and consequently, assure dynamical identity of the original and approximate problems. As known, the information about the internal symmetries of the system is contained by the Poisson brackets. The Hamiltonian of the system, in this sense, is the secondary quantity which fixes the hypersurface in the phase space of dynamic variables. The dynamic trajectory of the system lies on this hypersurface. Thus, to prevent losses or distortions of the internal symmetry properties, only the approximations that preserve the Poisson brackets should be selected. Only the Hamiltonian is subjected to the approximation procedure.

Many interesting problems of classical hydrodynamics possess certain spatial symmetry. For this reason, using conventional curvilinear coordinate systems may be more natural than using the Cartesian one. For instance, an evolution of an axially symmetrical jet is natural to be described in cylindrical, i.e., curvilinear coordinates. Also it is more comfortable sometimes to work not with the density and velocity as field variables, but with the density ρ and vorticity field $\boldsymbol{\gamma}=\mathrm{curl}\,\boldsymbol{\pi} \equiv \boldsymbol{\nabla} \times \boldsymbol{\pi}$.

Let us summarize that the system of Poisson brackets which describes the motions of inhomogeneous incompressible fluid is formulated in terms of density field ρ and vorticity field $\boldsymbol{\gamma}=\mathrm{curl}\,\boldsymbol{\pi}$, has the following form in the system of arbitrary curvilinear coordinates $\boldsymbol{\zeta}=(\zeta^1,\zeta^2,\zeta^3)$:

$$\{\varrho, \varrho'\} = 0, \quad (4)$$

$$\{\varrho, \gamma'^k\} = e^{knm} g^{-1/2} \partial_m \varrho \partial_n g^{-1/2} \delta, \quad (5)$$

$$\{\gamma^j, \gamma'^k\} = e^{ipj} e_{j\,\mathrm{ln}} e^{kmn} g^{-1/2} \partial_p \gamma^l \partial_m g^{-1/2} \delta. \quad (6)$$

For the details of derivations we refer the reader to works [15,16] and Appendixes C and D. Hereinafter, summation over repeated indices is implied. Additionally, $\delta=\delta(\boldsymbol{\zeta}-\boldsymbol{\zeta}')$ is the Dirac delta function, g is the determinant of the metric tensor, $\partial_i=\partial/\partial\zeta^i$ is the differentiation operator, $e^{ikj}\equiv e_{ikj}$ is the unit antisymmetric tensor (Levi-Civita symbol), and γ^j are the contravariant components [21] of the vector γ, that are related to the covariant components v_n of the hydrodynamic velocity as

$$\gamma^j = g^{-1/2} e^{ikn} \partial_k \varrho v_n. \quad (7)$$

The Poisson brackets given by Eqs. (4)–(6) present the initial point for the construction of various versions of the Hamiltonian contour dynamics in two-dimensional streams.

It is convenient to study such streams in the corresponding system of curvilinear coordinates with the following properties: ζ^3 coordinate lines are directed along the vortex lines, and ζ^1 and ζ^2 coordinate lines lie on $\zeta^3=\mathrm{const}$ stationary surfaces and form the system of surface coordinates. In such a coordinate system, we suppose that all fields are functions of only the surface coordinates $\boldsymbol{\zeta}=(\zeta^1,\zeta^2)$, and that vector vorticity field $\boldsymbol{\gamma}$ has only one component $\gamma^3=\gamma$, which according to Eq. (7) satisfies relation

$$\gamma = e^{\alpha\lambda} g^{-1/2} \partial_\alpha \varrho v_\lambda, \quad (8)$$

where $e^{\alpha\lambda}$ is the unit antisymmetric tensor.

In terms of the scalar fields, γ and ϱ, the Poisson brackets given by Eqs. (4)–(6) are reduced to the simpler form (see Appendix B)

$$\{\varrho, \varrho'\} = 0, \quad (9)$$

$$\{\varrho, \gamma'\} = e^{\lambda\alpha} g^{-1/2} \partial_\alpha \varrho \partial_\lambda g^{-1/2} \delta, \quad (10)$$

$$\{\gamma, \gamma'\} = e^{\lambda\alpha} g^{-1/2} \partial_\alpha \gamma \partial_\lambda g^{-1/2} \delta. \quad (11)$$

In publications [8–14] (see also references therein, and the Appendixes of this article) it was noted that certain important patterns of large-scale two-dimensional dynamics of an incompressible, inviscid fluid can be modeled as patches of constant vorticity and density. In this case, the description of vortex evolution can be reduced to the description of dynamics of discontinuity boundaries, or *contours*, while the rest of fluid can be ignored. Despite the fact that this approximation seems too strict from a physical viewpoint, comparisons between numerical contour dynamics and conventional numerical simulations have shown surprisingly good agreement with flows modeled by distributed vorticity at very high Reynolds number (see [8–14]). It appears that many general aspects of quasi-inviscid flows can be reproduced using the contour dynamics with a moderate number of vorticity and density levels. One of the advantages of this approach is the significant reduction of the number of dynamic variables and the running time needed for numerical simulations. Because contour dynamics equations are strongly nonlinear and nonlocal, a numerical approach to their solving is required. Analytical versions of contour dynamics have been much more narrowly used because their effectiveness depends greatly on the choice of dynamical variables. Also, the method of contour parameterization is frequently nontrivial and requires special considerations. It is worth pointing out that here the abilities of traditional formulations are much more limited. The requirement of physical obviousness, which is common in traditional formulations when selecting dynamical variables, does not guarantee the visual simplicity of the evolution of these variables in the phase space. Consequently, the methodology for formulating the problem should be flexible—on the one hand, permitting easy transitions from one phase space to another, and on the other hand, providing

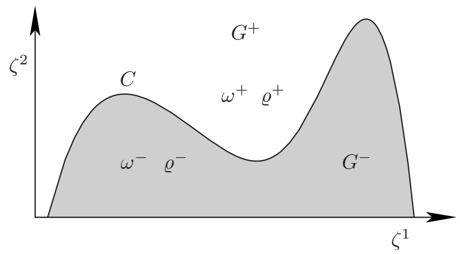

FIG. 1. Model of the equally vortexed regions with the free boundary.

easy controls when selecting models to be used in asymptotic approximations. The Hamiltonian version of the contour dynamics satisfies all these requirements.

Various Hamiltonian versions of contour dynamics for plane and axially symmetric models have been developed and discussed in the works [15–18].

Let us formulate the key parameters of the model.

For simplicity, let us consider (see Fig. 1) two regions G^+ and G^- that are separated by interface C on both sides of which density ρ and vorticity $\omega = g^{-1/2}(\partial_1 v_2 - \partial_2 v_1)$ attain constant values. Using signs $+$ and $-$ for the variables in regions G^+ and G^-, respectively, we write the density and covariant components of the hydrodynamic momentum in the form of decompositions

$$\varrho = \varrho^+ \theta^+ + \varrho^- \theta^-, \tag{12}$$

$$\varrho v_\alpha = \varrho^+ v_\alpha^+ \theta^+ + \varrho^- v_\alpha^- \theta^-. \tag{13}$$

Here, θ^+ and θ^- are mutually complimentary substantive characteristic functions

$$\theta^\pm = 1, \quad \text{if } \zeta \in G^\pm,$$

$$\theta^\pm = 0, \quad \text{if } \zeta \notin G^\pm,$$

which satisfy relations

$$\theta^+ + \theta^- = 1, \quad \theta^+ \theta^- = 0, \quad \partial_t \theta^\pm + v^\alpha \partial_\alpha \theta^\pm = 0.$$

The substitution of Eq. (13) into Eq. (8) leads to expression

$$\gamma = \omega^+ \varrho^+ \theta^+ + \omega^- \varrho^- \theta^- + \beta, \tag{14}$$

where

$$\beta = e^{\alpha\lambda} g^{-1/2} (\varrho^+ v_\lambda^+ - \varrho^- v_\lambda^-) \partial_\alpha \theta^+ \tag{15}$$

describes the part of momentum vorticity density that is concentrated on the contour. It is easy to see that even when velocity distribution is continuous, $v_\lambda^+ = v_\lambda^-$ at the contour, and value $\beta \neq 0$ if density jumps at the contour.

B. Axisymmetric stream

In the following discussion we consider only axisymmetric equally vortexed streams without a swirl (i.e., fluid rotation around the axis is absent). The simplest example of such streams is a round jet with a parabolic velocity profile. Hills vortex [14,22] is another less trivial example.

The natural coordinates for studying axisymmetric two-dimensional streams are the cylindrical coordinates, x, r, and φ; axial coordinate x and radial coordinate r serve as the surface coordinates ζ^1 and ζ^2, whereas φ is the azimuth coordinate whose coordinate lines coincide in direction with the vortex lines.

Assuming that the interface is specified by $r = \eta(x,t)$ (where η describes the shape of the interface), we take the substantive characteristic function θ^+ in the form of Heaviside step function

$$\theta^+ = \theta(r - \eta),$$

i.e., $\theta(z) = 1$ for $z \geqslant 1$ and $\theta(z) = 0$ for $z < 0$. Because of this choice Eqs. (12) and (14) can be rewritten in the form

$$\varrho = \varrho^- + (\varrho^+ - \varrho^-)\theta(r - \eta), \tag{16}$$

$$\gamma = \omega^- \varrho^- + \nu\theta(r - \eta) + \mu r^{-1} \delta(\eta - r), \tag{17}$$

where $\nu = \varrho^+ \omega^+ - \varrho^- \omega^-$ and μ is the jump of the tangential momentum component on the contour; i.e.,

$$\mu = -[\eta_x(\varrho^+ v_2^+ - \varrho^- v_2^-) + (\varrho^+ v_1^+ - \varrho^- v_1^-)]_{r=\eta}. \tag{18}$$

Relations (16) and (17) allow us to easily recalculate the Poisson brackets for variables (γ, ϱ) to the Poisson brackets for variables (η, μ) specified on the contour. After substituting Eqs. (16) and (17) into the system of Eqs. (9)–(11) and solving it under the assumption that $\varrho^+ - \varrho^- \neq 0$, i.e., in the presence of the density jump, we obtain

$$\{\eta, \eta'\} = 0, \tag{19}$$

$$\eta\{\eta, \mu'\} = -\partial_x \delta(x - x'), \tag{20}$$

$$\{\mu, \mu'\} = -\nu \partial_x \delta(x - x'). \tag{21}$$

If $\nu \neq 0$, transformation

$$\xi = \mu - \nu\eta^2/2, \tag{22}$$

reduces the brackets given by Eqs. (19)–(21) to the brackets

$$\{\xi, \mu'\} = 0, \tag{23}$$

$$\{\xi, \xi'\} = \nu \partial_x \delta(x - x'), \tag{24}$$

$$\{\mu, \mu'\} = -\nu \partial_x \delta(x - x'). \tag{25}$$

These brackets correspond to the equations of motion

$$\partial_t \xi = \nu \partial_x \frac{\delta H}{\delta \xi}, \quad \partial_t \mu = -\nu \partial_x \frac{\delta H}{\delta \mu}, \tag{26}$$

which conserve not only the Hamiltonian H, but also the integrals

$$P = \frac{1}{2} \int dx (\xi^2 - \mu^2), \tag{27}$$

$$T_1 = \int dx\, \xi, \quad T_2 = \int dx\, \mu. \tag{28}$$

If the conservation of the integral given by Eq. (27), which serves as the vortex momentum, is caused by the translational invariance of the Hamiltonian H, then T_1 and T_2 are annihilators of brackets (23)–(25) and, hence, are conserved for an arbitrary Hamiltonian H. This means that the Hamiltonian in such models is determined up to linear terms that do not affect the equations of motion.

Now let us express Hamiltonian H in terms of variables ξ and μ by considering the boundary value problem. For axisymmetric streams of incompressible fluid that are considered here, this problem is formulated in terms of the Stokes stream function, $\psi=\psi^+\theta^+ + \psi^-\theta^-$.

The incompressibility condition, described as div $\mathbf{v}=0$ or as $\partial_x v_1 + r^{-1}\partial_r r v_2 = 0$ in cylindrical coordinates, allows us to introduce stream function ψ. After taking into account relations

$$v_1 = -r^{-1}\partial_r\psi, \quad v_2 = r^{-1}\partial_x\psi, \tag{29}$$

which relate the corresponding velocity components to the stream function ψ, and using Eqs. (8) and (18), we arrive at the boundary value problem

$$r^{-1}(\partial_x r^{-1}\partial_x + \partial_r r^{-1}\partial_r)\psi^{\pm} = \omega^{\pm}, \tag{30}$$

$$[(\eta_x\partial_x - \partial_r)(\varrho^-\psi^- - \varrho^+\psi^+)]_{r=\eta} = \eta\mu. \tag{31}$$

The first expression, Eq. (30), is the azimuthal covariant component of vorticity which is supposed to be constant. The stream function is bounded on axis $r=0$. The boundary condition in Eq. (31) follows from the existence of the vortex sheet of intensity μ on the boundary $\eta(x,t)$. Let us note that the existence of the vortex sheet is an imperative attribute of contact frontier dynamics, i.e., $\mu\not\equiv 0$ when the boundary separate two fluids with different density.

When necessary, this boundary value problem is supplemented by the conditions of stream function continuity at the inner interfaces and its constancy at the outer interfaces.

III. AXISYMMETRIC MODEL WITH FREE BOUNDARY

Since the presence of free boundary is equivalent to condition $\varrho^+=0$, we will consider the fluid with unit density $\varrho^-=1$ and vorticity $\omega^-=\omega$ and present only inside the region $G^-=G$. Similar models of streams, but without density jumps, were considered in Ref. [17,18]. Assuming that the problem contains characteristic spatial scale L, we introduce corresponding time scale $T=(\omega L)^{-1}$ and dimensionless dependent and independent variables.

Here, we shall restrict our study to axially symmetric vorticity-homogeneous flows with free boundary only. In other words, it is supposed (see Fig. 2) that the fluid of unit density is concentrated only inside domain G restricted by contour C on which the unit jump of density takes place.

Then the equations of motion (26) take form

$$\partial_t\xi = -\partial_x\frac{\delta H}{\delta\xi}, \quad \partial_t\mu = \partial_x\frac{\delta H}{\delta\mu}. \tag{32}$$

When calculating the Hamiltonian H, we start from the kinetic energy integral

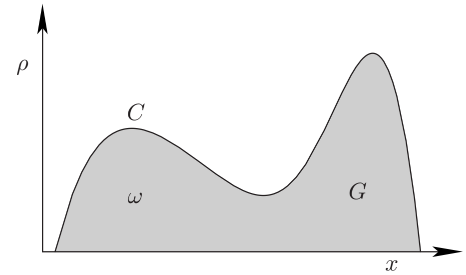

FIG. 2. Sketch of the axisymmetric equally vortexed stream with the free boundary.

$$\mathcal{E} = \frac{1}{2}\int_0^{\eta} dr\, r^{-1}((\partial_r\psi)^2 + (\partial_x\psi)^2)$$
$$= -\frac{1}{2}\int dx\, dr(r\theta(\eta - r)\psi + \mu\psi\delta(\eta - r)) + \cdots = H + \cdots \tag{33}$$

with H matches up to terms that do not affect the equations of motion (32). For this reason, the Hamiltonian H can have any sign, even though $\mathcal{E}$ is a positive finite quantity.

According to Eqs. (30) and (31), the following boundary value problem exists for the stream function ψ^- (in terms of which the kinetic energy $\mathcal{E}$ is expressed):

$$(\partial_x r^{-1}\partial_x + \partial_r r^{-1}\partial_r)\psi^- = r, \tag{34}$$

$$(\eta_x\partial_x - \partial_r)\,\psi^-|_{r=\eta} = \eta\mu, \tag{35}$$

which is supplemented by the *gauge* condition on the axis, $\psi^-|_{r=0}=0$.

Hereinafter we use the method of pseudo-differential operators [17–20].

Following this method, a general solution of the boundary value problem specified by Eqs. (34) and (35) is sought in an *operator* form

$$\psi(x,r,t) = \frac{1}{8}r^4 + rI_1(r\Gamma)A(x,t). \tag{36}$$

The structure of Eq. (34) serves as a prompt. The first term in Eq. (36) is a partial solution to Eq. (34): It describes a nonperturbed state of the equally vortexed jet. The second term describes the perturbation of the jet. Here, $A=A(x,t)$ is a function of time t and axial spatial coordinate x; expression $I_1(r\Gamma)$ is the modified Bessel's function of the first order defined by the standard expression $I_s(r\Gamma)=i^{-s}J_s(ir\Gamma)$. The argument of the modified Bessel's operator function in Eq. (36) is the product of the transversal coordinate r on the *operator* Γ. This operator defined by expression $\Gamma=\partial_x\hat{H}$, where operator $\hat{H}$ is the so-called Hilbert transformation:

$$\hat{H}f(x) = \frac{1}{\pi}\int dx'\frac{f(x')}{x - x'}.$$

The remarkable properties of the operators $\hat{H}$, and Γ are the identities $\hat{H}^2=-1$, $\hat{H}^{+}=-\hat{H}$, $\Gamma^2=-\partial_x^2$, and $\Gamma^{+}=\Gamma$. It means that operator $\hat{H}$, when applied twice to a function of x, multiplies the function by minus 1, and that operator Γ applied to e^{ikx} multiplies the exponent by $|k|$, and so forth. Therefore, more complex operator combinations can be constructed and used.

Let us restate briefly some useful properties (for more details, see Ref. [18]).

The action of the pseudo-differential operator $F(r,\Gamma)$ on a function $f(x)$ is equivalent to the integral transformation

$$F(r,\Gamma)f(x)=\frac{1}{2\pi}\int dx_1 dk\, F(r,|k|)e^{ik(x_1-x)}f(x_1) \qquad (37)$$

and is unambiguously determined by the function $F(r,|k|)$ called the symbol of the pseudo-differential operator.

If the operator function $F[\eta(x),\Gamma]$ depends on two noncommutative operators, it is necessary to follow the ordering rule: the operators act in the order from the right to the left. This rule allows us to distinguish conjugated operators $F[\eta(x),\Gamma]$ and $F[\Gamma,\eta(x)\equiv F^{+}(\eta(x)\ ,\Gamma]$. Operator $F(\eta(x),\Gamma)$ indicates that the first to act is operator Γ, and the operator of multiplication by $\eta(x)$ acts second. Operator $F(\Gamma,\eta)$ implies conversely that the operator of multiplication by $\eta(x)$ acts first, and operator Γ acts second. For this reason, it should be noted that operators $F[\eta(x),\Gamma]$ and $F^{+}[\eta(x),\Gamma]$ are characterized by different integral transformations

$$F[\eta(x),\Gamma]f(x)=\frac{1}{2\pi}\int dx_1 dk\, F[\eta(x),|k|]e^{ik(x_1-x)}f(x_1), \qquad (38)$$

$$F[\Gamma,\eta(x)]f(x)=\frac{1}{2\pi}\int dx_1 dk\, F(\eta(x_1),|k|)e^{ik(x_1-x)}f(x_1). \qquad (39)$$

The advantage of the method of pseudo-differential operators is that operator functions can be transformed and simplified in analytical calculations as regular functions. For example, operator functions can be developed in the perturbation series with respect to their operator arguments.

Now let us return to our key derivations.

Direct substitution of expression (36) into the boundary condition given by Eqs. (35) yields the following equation with respect to function $A(x,t)$

$$\partial_x I_0(\eta\Gamma)\hat{H}A=-\xi. \qquad (40)$$

After some algebra and when taking into account Eqs. (36), (22), and (40), Hamiltonian (33) takes form

$$H=-\frac{1}{2}\int dx\left(\frac{1}{2\times 4!}\eta^4(\eta^2-3)+\frac{1}{8}\mu\eta^2(\eta^2-2)+\eta^2 I_2(\eta\Gamma)\frac{1}{\Gamma}A+\mu\eta I_1(\eta\Gamma)A\right). \qquad (41)$$

Here, we omit the terms that do not affect the equations of motion.

It should be reminded that the shape of surface $\eta(x,t)$ and function $A(x,t)$ must be expressed in the Hamiltonian via variables $\xi(x,t)$ and $\mu(x,t)$. Hamiltonian (41) contains two types of terms: Those that are determined by the magnitude of perturbation η and those that depend on the spatial derivative $\hat{\partial}_x$ folded in operator Γ.

The simplification that follows will be based on the additional assumptions about the relationship between nonlinear (η) and dispersion $(\hat{\partial}_x)$ effects. Let us assume that the considered perturbations are such that the contribution of the terms with spatial derivatives of higher orders may be neglected. The small parameter used to form the series is the ratio of the perturbation magnitude (which is comparable to the jet radius) to the characteristic x scale of the perturbation. This approximation is known as the nonlinear dispersion approximation [23,24].

We will seek the solution of pseudo-differential equation (40) as a power series with variable Γ.

In the framework of the "minimal" model which intends to provide only a qualitative description, for the primary approximation it is sufficient to take into account only the terms of the second order with respect to derivative ∂_x.

In this case, by expanding the pseudo-differential operator $I_0(\eta\Gamma)$ into the perturbation series $I_0(\eta\Gamma)=1+\frac{1}{4}\eta^2\Gamma^2+\cdots\equiv 1-\frac{1}{4}\eta^2\partial_x^2+\cdots$ and by capturing only the terms of the second order with respect to Γ, one can easily obtain the solution of Eq. (40) in the form

$$\Gamma A=-\left(1+\frac{1}{4}\partial_x\eta^2\partial_x+\cdots\right)\xi$$

and then the leading part of the Hamiltonian (41):

$$H=\int dx\left(\frac{1}{3!}(\xi^3-\mu^3)-\frac{3}{4!}(\xi-\mu)^2\xi_x^2\right). \qquad (42)$$

Here we omitted the fifth order terms with respect to the fields and the terms containing higher orders of derivatives. The Hamiltonian H is fixed by the initial conditions and its value can be either positive, or negative.

The corresponding equations of contour dynamics are derived using Eq. (32) and have the following form in this approximation

$$\partial_t\xi=-\partial_x\frac{\delta H}{\delta\xi}=-\frac{1}{2}\partial_x\left(\xi^2-\frac{1}{2}(\xi-\mu)\xi_x^2+\frac{1}{2}\partial_x(\xi-\mu)^2\xi_x\right), \qquad (43)$$

$$\partial_t\mu=\partial_x\frac{\delta H}{\delta\mu}=-\frac{1}{2}\partial_x\left(\mu^2-\frac{1}{2}(\xi-\mu)\xi_x^2\right). \qquad (44)$$

The obtained equations conserve the Hamiltonian H and the integrals

$$T_1=\int dx\,\xi,\quad T_2=\int dx\,\mu,$$

and

$$P=\frac{1}{2}\int dx(\xi^2-\mu^2). \qquad (45)$$

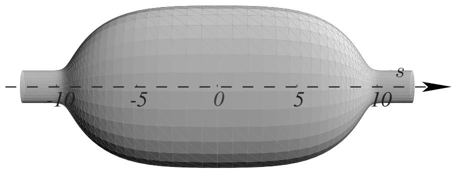

FIG. 3. Solitonlike perturbation on a uniformly whirling free jet.

IV. NUMERICAL TEST

We numerically simulated the evolution of the initial perturbation [described by Eqs. (43) and (44)] during the formation process of the localized structure, but before nonlinear instability arises (see discussion below). We chose the following initial conditions for the surface: $\eta=\sqrt{2(\xi-\mu)}$: $\xi(x,0)=(1/2)\{1+2\exp[-(x^2/36)]\}$, $\mu(x,0)=0$. The results of the test show that the initial perturbation breaks down as a result of *joint effects of nonlinearity and dispersion* on the sequence of localized structures (similar to *solitons*) moving with individual constant velocities. After a while, each of the impulses moves as a separate localized wave with its own velocity defined by the maximum of its amplitude.

The fact that such quasilimited states are possible allows us to consider a special class of traveling waves.

V. SOLITONS AND COMPACTONS

Let us consider the analytical solutions for Eqs. (43) and (44) in the form of a traveling wave

$$\xi=\xi(s),\quad \mu=\mu(s),\quad s=x-ct,$$

propagating with constant velocity c without profile deformation. After the substitution of these solutions into Eqs. (43) and (44) and the subsequent integration and simple algebra, these equations attain the following form:

$$\frac{\xi^3-\mu^3}{3}+\frac{1}{4}(\xi-\mu)^2\xi_s^2-c(\xi^2-\mu^2)+c_1\xi-c_2\mu-c_3=0, \tag{46}$$

$$\mu^2-\frac{1}{2}(\xi-\mu)\xi_s^2-2c\mu+c_2=0. \tag{47}$$

Here, integration constants c_1, c_2 and c_3 are specified by the type of the solution determined by the behavior of ξ and μ at

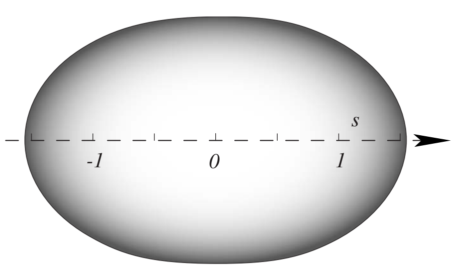

FIG. 4. Compacton for a model of a vorticity-homogeneous jet.

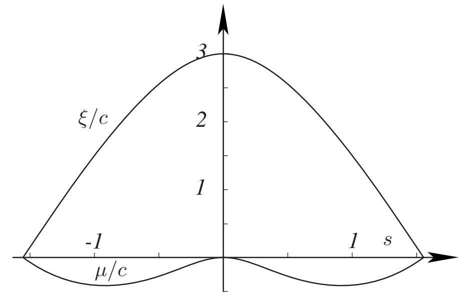

FIG. 5. Plots of functions ξ/c and μ/c and contour η.

$s\to\pm\infty$. Equations (46) and (47) can have either periodic, or localized solutions.

In order to study the final development stage of a strongly disturbed jet stream, such as when the jet splits into separate vortex blobs, the solutions with compact support are the most interesting among all solutions.

Solutions of the classical soliton type—which exponentially decrease for $s\to\pm\infty$ and appear in the regime of unperturbed jet when $\eta|_{s=\infty}=1$ and $\mu|_{s=\infty}=0$—are realized for the choice of constants

$$c_1=\frac{1+2a^2}{12},\quad c_2=0,\quad c_3=\frac{a^2}{24},\quad c=\frac{2+a^2}{6}.$$

Here, parameter a is the soliton amplitude. The image of the jet perturbed by the soliton solution is presented in Fig. 3.

In contrast to the traditional soliton solutions, the disturbances called compactons [24] do not have extended "tails" vanishing only at infinity, and as a result, do not retain the memory about the undisturbed regime. In the framework of the considered model, the existence of compactons can be interpreted as the final stage of development of a strongly nonlinear perturbation. At this stage, the jet breaks into individual vortex blobs. For this reason, in the framework of the considered model, compactons can serve as structures dominant at the disintegration stage of the evolution of the jet stream. The corresponding class of solutions to Eqs. (46) and (47) is realized with parameters $c_1=c_2=c_3=0$. Equations (46) and (47) allow for solutions *without* exponential "tails," such as distributions similar to Fig. 4. In this specific case, Eqs. (46) and (47) use parametrization

$$\xi=c(1-\cos\varphi+2\sin\varphi),\quad \mu=c(1-\cos\varphi-\sin\varphi), \tag{48}$$

which for $0\leqslant\varphi\leqslant\pi/2$ and, respectively, $c\geqslant 0$, leads to equation

$$\frac{3}{4}c(\sin\varphi+2\cos\varphi)^2\varphi_s^2=\cos\varphi. \tag{49}$$

The behavior of functions ξ and μ is graphically presented in Fig. 5.

By virtue of Eq. (22), relationship $2(\xi-\mu)=\eta^2$ determines the contour profile. The compacton shape can consequently be determined from the simple relation

$$\eta^2=6c\sin\varphi. \tag{50}$$

Using this formula, one can easily conclude that, if the transverse half-size of the compacton R is taken as the length scale, in the system with scale units $L=R$ and $T=\omega R c^{-1}$, the dimensionless velocity of the compacton is constant, $c=1/6$. The dimensional compacton velocity v is obtained from formula $v=cL/T$ which yields

$$v=\frac{\omega}{6}R^2,$$

where the transverse half-size of the compacton R serves as the controlling parameter.

By suggesting that the maximum value $\eta=1$ is reached at the point $s=0$ for $\varphi=\pi/2$, we arrive from Eqs. (49) and (50) to the parametric description of the compacton shape in the form

$$\eta=\sin^{1/2}\varphi,$$

$$s=\pm\sqrt{2}\left[\frac{1}{2}\cos^{1/2}\varphi+E\left(\frac{\pi}{4},2\right)-E\left(\frac{\varphi}{2},2\right)\right], \tag{51}$$

where $E(\varphi,k)$ is the elliptic integral of the second kind.

VI. INSTABILITY OF THE COMPACTONS

Let us analyze whether the obtained solutions are stable. This question is of particular significance because it has been observed that localized disturbances sometimes collapse, i.e., form a singularity within a finite time [25,26]. This phenomenon can be viewed as a permanent self-constriction of a localized disturbance which must be accompanied by the infinite increase in its amplitude at fixed integrals of motion.

To analyze the instability, we introduce new variables

$$q=\xi-\mu,\quad z=\frac{1}{2}(\xi+\mu),$$

in terms of which the equations of contour dynamics (43) and (44), the Hamiltonian H, and the integral of motion P (the latter one is particularly important) take the following form, which is more convenient for further analysis:

$$\partial_t q=-\partial_x\frac{\delta H}{\delta z},\quad \partial_t z=-\partial_x\frac{\delta H}{\delta q}, \tag{52}$$

$$H=I_1-I_2 \tag{53}$$

$$I_1=\frac{1}{4!}\int dx(q^3+12qz^2),$$

$$I_2=\frac{3}{4!}\int dx\, q^2\left(\frac{1}{2}q_x+z_x\right)^2,$$

$$P=\int dx\, qz. \tag{54}$$

Here, z_x and q_x are the derivatives with respect to argument x. The equations of motion (52) for the compacton solutions in the form of a traveling wave $q=q(s)$, $z=z(s)$, $s=x-ct$, propagating with constant velocity c without profile deformation, are written in the form

$$cq-\frac{\delta H}{\delta z}=0,\quad cz-\frac{\delta H}{\delta q}=0,$$

which can be treated as the consequence of the variational problem

$$\delta(H-cP)=0, \tag{55}$$

where the compacton velocity c serves as a Lagrange multiplier.

In the context of Eq. (55), the compacton solution is an equilibrium point in the infinite-dimensional (with respect to parameter x) phase space of fields q and z. To determine the type of this point (center, saddle, …) in the simplest way, let us consider such a variation of the compacton solution $q\rightarrow q(s,\alpha,\ldots)$ that depends on the *finite* number of continuous parameters $(\alpha,\ldots)$. and conserves the integral of motion P. The transformation $q\rightarrow q(s,\alpha,\ldots)$ of the compacton solution makes Hamiltonian H an ordinary function of the parameters, $H\rightarrow H(\alpha,\ldots)$. According to the general Hamiltonian method, parameters $\alpha,\ldots$. can serve as the generalized coordinates with respect to which the variations in Eq. (55) are accomplished. Our intent is to see if the stationary point is a saddle point in the finite-dimensional phase space of these generalized coordinates $(\alpha,\ldots)$. If this is the case, the corresponding compacton solution is unstable.

As one of such possible transformations, we consider the transformation

$$q(s)\rightarrow\frac{\beta}{\sqrt{\alpha}}q\left(\frac{s}{\alpha}\right),\quad z(s)\rightarrow\frac{1}{\beta\sqrt{\alpha}}z\left(\frac{s}{\alpha}\right),$$

where α and β are the parameters of this transformation. In this case $H\rightarrow H(\alpha,\beta)$ on the compacton solution.

First, we consider the case with $\beta=1$. Applying the transformation to the Hamiltonian H yields

$$H(\alpha,1)=\alpha^{-1/2}I_1-\alpha^{-3}I_2. \tag{56}$$

Integrals I_1 and I_2 on the compacton solutions are related via expressions

$$I_1+I_2=cP,\quad \frac{1}{2}I_1-3I_2=0, \tag{57}$$

which follow from Eq. (46) and from the equilibrium condition $\partial_\alpha H(\alpha)=0$, respectively. Therefore, Hamiltonian $H(\alpha,1)$ has the form

$$H(\alpha,1)=\frac{cP}{7}[6\alpha^{-1/2}-\alpha^{-3}], \tag{58}$$

where cP is positive according to the relation $cP=7I_2$. The plot of this function is shown in Fig. 6 (line a). Function $H(\alpha)$ has no lower bound for $\alpha\rightarrow 0$, vanishes for $\alpha\rightarrow\infty$, and reaches its maximum at the equilibrium point $\alpha=1$.

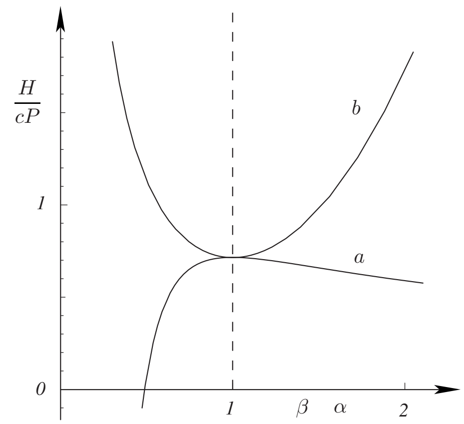

FIG. 6. Compacton Hamiltonian as a function of the transformation parameters (*a*) α and (*b*) β.

The case with $\alpha=1$ can be analyzed similarly. The plot of function $H(1,\beta)\equiv H(\beta)$ is shown in Fig. 6 (line *b*). Function $H(\beta)$ has no upper bound for $\beta\to 0$ and $\beta\to\infty$ and reaches its minimum at the equilibrium point $\beta=1$.

Thus we determined that the compacton solution is a saddle point with Hamiltonian $H(\alpha,\beta)$, and therefore, this solution is unstable. If the solution is unstable, the compacton can collapse. In the next section we study such a collapse.

VII. COLLAPSE GENERATED BY AN ASYMMETRICAL MODE IN THE VORTEX SHEET DISTRIBUTION

Let us consider here a collapse generated by an *asymmetrical* mode in the vortex sheet distribution. In order to qualitatively study the evolution of the instability, we construct a sample model that can be solved exactly. The deduction of such models has been conceptually formulated by Fermi in works [3–5]. The approach is based on the method of generalized coordinates which takes account of the advances of the variational principle.

The simplest way to formulate this principle for Eqs. (52) is to assign variable q as a "generalized coordinate" and introduce the so-called "generalized impulse" p using expression $p_x=z$. In terms of variables q and p, the equations of motion become canonical, and the variational principle is formulated in a standard way. The action is

$$S=\int dt\,\mathcal{L},$$

where the Lagrangian $\mathcal{L}$ is defined as

$$\mathcal{L}=-H+\int dx\, q\partial_t p. \tag{59}$$

The next step is to parameterize the generalized coordinate and impulse using the so-called test functions. Following the method described in the previous section, we employ the two-parameter transformation that conserve the integral of motion P. We consider a combination of scale and gradient transformations [29]

$$q\to\alpha^{-1}q\left(\frac{x}{\alpha}\right),\quad z\to z\left(\frac{x}{\alpha}\right)+\varkappa\alpha\partial_x q\left(\frac{x}{\alpha}\right). \tag{60}$$

Here, α and $\varkappa$ are parameters of the transformations.

The choice of transformation (60) as the starting point for the parameterization is motivated by the following reasons. First, the transformation (60) leaves the integral of motion, P, invariant. Second, it does not change integral [30]

$$I=\int dx\,q=\frac{1}{2}\int dx\,\eta^2,$$

and finally, the gradient correction can be interpreted as an asymmetrical mode excitation at the free boundary of the vortex sheet. In essence, such a correction is based on the assumption that the fundamental cause of compacton instability is the presence of the vortex sheet which, as known [14], is characterized by an extremely high instability and has a tendency to lose smoothness and to roll up during the evolution.

Now, we suppose that the parameters of transformation, α and $\varkappa$, depend on time. We choose as the compacton solutions those functions of q, p that appear in the right-hand side of Eq. (60). The function on the left-hand side of Eq. (60) is called a trial function.

Substitution of the trial functions into Eqs. (59) and integration lead us to a discrete model with generalized coordinates α, $\varkappa$ and the Lagrangian [31]

$$\mathcal{L}=\frac{d\alpha}{dt}\varkappa-\mathcal{H},$$

where value $\mathcal{H}$ is playing the part of the Hamiltonian for this discrete model and is determined by expression

$$\mathcal{H}=\epsilon H(\alpha,\varkappa).$$

Here, $\epsilon^{-1}=\frac{1}{2}\int ds\,q^2$ is a numerical coefficient and $H(\alpha,\varkappa)$ is the Hamiltonian of the initial model calculated using the trial functions.

We calculate the Hamiltonian, using the following process. First we substitute compacton solution (48)–(51) into Eqs. (53) and (54). Because the solution is dimensionless, calculations of H, I_1, I_2, and P (i.e., the integrals of elliptical functions) are done numerically and produce numerical values for H, I_1, I_2, and P. Then we take expressions (60) and (48)–(51) and substitute them one more time into Eqs. (53) and (54). In this second substitution, the form of the transformation allows us to redefine the integration variables and take parameters α and $\varkappa$ outside the integration sign. By combining all expressions (including the calculated values H, I_1, I_2, and P), we can find that the Hamiltonian of the chosen discrete model, $\mathcal{H}$, is defined by the expression

$$\mathcal{H}=(0.213-0.053\alpha^{-3})\varkappa^2+(0.221\alpha^{-2}-0.040\alpha^{-3}-0.052\alpha^{-4}-0.023\alpha^{-5}).$$

The leading terms which define the dynamical system behavior *near* the point of collapse $t=t_0$ are the first two terms. For this reason, to study the system qualitatively we consider a "truncated" Hamiltonian

$$\tilde{\mathcal{H}} \simeq \left(B - \frac{C}{\alpha^3}\right)\varkappa^2$$

for which the "truncated" equations of motion have the following form:

$$\frac{d\alpha}{dt} = \frac{\partial\tilde{\mathcal{H}}}{\partial\varkappa} = 2\varkappa\left(B - \frac{C}{\alpha^3}\right), \quad \frac{d\varkappa}{dt} = -\frac{\partial\tilde{\mathcal{H}}}{\partial\alpha} = -3C\frac{\varkappa^2}{\alpha^4}.$$

Since for $t \to t_0$, the leading terms have to balance, we obtain that $\varkappa \sim (t_0 - t)^{-1}$ and $\alpha \to \alpha_0$. The equality $B = C/\alpha_0^3$ is thus satisfied for $t \to t_0$. This also allows us to find the limit value of the parameter α (which is responsible for the scale transformation): $\alpha_0 = (C/B)^{1/3} = 0.630$. It is now possible to write the approximate equations for α near the point of collapse where $\alpha = \alpha_0 + \alpha'$ (α' represents a small perturbation)

$$\frac{d\alpha'}{dt} = 6B\alpha_0^{-1}\varkappa\alpha', \quad \frac{d\varkappa}{dt} = -3B\alpha_0^{-1}\varkappa^2.$$

The solutions, obviously, have the power dependency with respect to t as

$$\varkappa \sim (t_0 - t)^{-1}, \quad \alpha' \sim (t_0 - t)^2.$$

These solutions describe the collapse scenario generated by an asymmetrical mode on a free boundary of a vortex sheet.

VIII. SELF-SIMILAR COLLAPSE

In order to investigate the final stage of the development of the instability development, we consider a self-similar collapse scenario, suggesting that near the collapse point $t = t_0$, i.e., when $t \to t_0$, the leading terms are

$$q \simeq Q(x - ct), \quad z \simeq (t_0 - t)^{-1}Z(x - ct). \tag{61}$$

Here, c is the disturbance propagation velocity. The solutions given by Eqs. (61) describe the collapse regime generated at the free boundary by the asymmetric mode in the vortex sheet distribution.

Let us move into the comoving reference frame $s = x - ct$. Since this transition is accompanied by transformation $H \to H + cP$, the Hamiltonian given by Eq. (53) takes form

$$H = I_1 - I_2 + cP$$
$$= \frac{1}{2}\int ds\left[2cqz + qz^2 + \frac{1}{12}q^3 - \frac{1}{4}q^2\left(\frac{1}{2}q_s + z_s\right)^2\right]. \tag{62}$$

Here, z_s and q_s are the derivatives with respect to argument s. By assuming that near the collapse point, according to Eq. (61), the terms highest in order with respect to z make most contribution, and by omitting less significant terms in Eq. (62), we obtain the *truncated* Hamiltonian

$$H_{\mathrm{tr}} \simeq \frac{1}{2}\int ds\left[qz^2 - \frac{1}{4}q^2 z_s^2\right].$$

The equations of motion corresponding to the Hamiltonian H_{tr} have form

$$\partial_t q = -\partial_s \frac{\delta H_{\mathrm{tr}}}{\delta z} = -\partial_s\left[qz + \frac{1}{4}\partial_s(q^2 z_s)\right], \tag{63}$$

$$\partial_t z = \partial_s \frac{\delta H_{\mathrm{tr}}}{\delta q} = \partial_s\left[\frac{1}{2}z^2 - \frac{1}{4}qz_s^2\right]. \tag{64}$$

Substitution of Eq. (61) into Eqs. (63) and (64) yields the following equations for the structure functions Q and Z:

$$QZ \simeq -\frac{1}{4}\partial_s(Q^2 Z_s), \tag{65}$$

$$Z(1 + Z_s) \simeq \frac{1}{4}\partial_s[QZ_s^2]. \tag{66}$$

These equations produce the integral

$$Q^2 Z_s(3Z_s + 2) = C,$$

and, consequently, their only regular solution is

$$Q = b^2 - s^2, \quad Z = -\frac{2}{3}s, \tag{67}$$

which corresponds to the choice $C = 0$ and is localized in the interval $-b \leqslant s \leqslant +b$.

Discussion of other (singular) solutions, which can also be of interest, is beyond the scope of this work.

It is worth noting that conditions $H_{\mathrm{tr}} = 0$, $P = 0$, and $T_1 = -T_2 = \mathrm{const}$ are automatically valid on self-similar expression (61). Thus, these self-similar solutions do not violate the conservation laws and do correspond to the weak-collapse scenario.

The structure function Q characterizes the shape of the disturbance, and thus it characterizes the distribution of the self-similar component of the vortex sheet. It is easily seen from Eq. (67) that the disturbance at the selfsimilar stage is an oblate ellipsoid with the ratio of semiaxes $a/b = \sqrt{2}$. The semiaxes of the ellipsoid (longitudinal b and transversal a) can be calculated using the conservation law

$$I = \int dx\, q = \frac{4}{3}b^3.$$

In particular, if the initial state is a compacton, $I = 1.09$ and, therefore, $b = 0.93$ and $a = 1.32$. The asymmetric mode of the vortex sheet is intensified with time on the surface of this ellipsoid according to the law $(t_0 - t)^{-1}$ and with distribution $z = -\frac{2}{3}s$.

IX. CONCLUSION

Hydrodynamic mixing in stratified media is a complex process. We analyzed the general picture of fluid evolution by applying the Hamiltonian approach to a system with a continual number of degrees of freedom. We formulated the Hamiltonian version of contour dynamics for a model of axially symmetric, equally vortexed jet streams with a free boundary. In order to grasp the tendency of the processes, we parametrized (in the spirit of the idea of Fermi) the exact

Hamiltonian by a finite number of trial parameters (generalized coordinates). In particular, we studied the evolution and collapse of dominant structures (compactons) which appear in strongly perturbed jet streams at the stage of their decay.

Our study showed that the evolution of the equally vortexed jet streams with the Atwood number close to 1 (large density contrast at the interface) may lead to breaking of the jet into separate vortex blobs (compactons). The approach presented above made it possible to determine the shape of these structures and to analyze the mechanism of their instability. We found that when compacton collapse occurred due to generation of an asymmetric mode on the vortex sheet, the compacton shape did not get distorted. The scale transformation parameter in the model stopped at the value $\alpha_0 = 0.630$ indicating that the transverse size of the compacton increased by a factor of 1.26, while the longitudinal contracted along the x axis by a factor of 1.59. No further change of compacton shape occurred. In essence, the collapse effect was condensed into an increase of intensity μ of the vortex sheet according to the law $\mu \sim (t_0 - t)^{-1}$. More detailed calculations should reveal the same tendency, but may be much more complex and subject to potential noise that should be carefully eliminated.

In conclusion, we want to emphasize that such simplified models should not be interpreted as producing the final answer. They are useful only as effective means for capturing the tendency of the processes. The key idea, as formulated by Fermi, is that the physics even of a complex hydrodynamical phenomenon can often be understood and described by a qualitative model with a small number of parameters.

ACKNOWLEDGMENTS

We are grateful to V.M. Ponomarev and E. P. Tito for stimulating discussions. This work was supported by the Russian Foundation for Basic Research (Project No. 06-05-64185), by the Presidium of the Russian Academy of Sciences (program Mathematical Methods in Nonlinear Dynamics), and by the Council of the President of the Russian Federation for Support of Leading Scientific Schools (Project No. NSh-4166.2006.5).

APPENDIX A: POISSON BRACKETS FOR AN INCOMPRESSIBLE NONUNIFORM EULERIAN FLUID

The equations of motion (in Cartesian coordinates) for a *nonuniform incompressible* ($\partial_j v_j = 0$) fluid are formulated in terms of the Eulerian variables: mass density ρ, velocity $\mathbf{v}$ and pressure p as

$$\partial_t v_i + v_k \partial_k v_i = -\frac{1}{\rho}\partial_i p + \frac{1}{\rho} f_i, \qquad \text{(A1)}$$

$$\partial_t \rho + v_k \partial_k \rho = 0, \qquad \text{(A2)}$$

$$\partial_k v_k = 0, \qquad \text{(A3)}$$

where $\mathbf{f}$ is the resultant of exterior forces that do not violate conservativeness of the fluid. This means that equations of motion (A1)–(A3) preserve the total energy, H, given by the sum of the kinetic, T, and potential, U, energy of the fluid given by

$$H = T + U,$$

$$T = \int d\mathbf{x} \rho \frac{\mathbf{v}^2}{2}, \quad U = U[\rho], \qquad \text{(A4)}$$

where U is, in general, an arbitrary functional of density ρ. For simplicity, we assume that the fluid is unbounded.

The evolution equation for the momentum density $\boldsymbol{\pi} = \rho \mathbf{v}$ can be found as follows. Using Eqs. (A1) and (A2) leads us to

$$\partial_t \pi_i + v_k(\partial_k \pi_i - \partial_i \pi_k) = -\partial_i\left(p + \rho\frac{\mathbf{v}^2}{2}\right) + \frac{\mathbf{v}^2}{2}\partial_i \rho + f_i. \qquad \text{(A5)}$$

By taking the curl of Eq. (A5), and thereby eliminating the gradient term with pressure, we obtain equation

$$\partial_t \gamma_i = e^{imn}\partial_m\left[e^{nkl} v_k \gamma_l - \frac{\mathbf{v}^2}{2}\partial_n \rho + f_n\right], \qquad \text{(A6)}$$

which describes the evolution law for the vorticity of momentum density $\boldsymbol{\gamma} = \boldsymbol{\nabla} \times \boldsymbol{\pi}$ under the action of exterior conservative forces.

We now show that the equations of motion for an incompressible inhomogeneous fluid reformulated in terms of the momentum density vorticity are described by the Hamiltonian with the local Poisson brackets $\{\gamma_i, \gamma_k'\}$ and $\{\rho, \gamma_k'\}$ [28,27].

First, we compute the Poisson bracket $\{\rho, \gamma_k'\}$. Because the model is expected to be Hamiltonian, we have every reason to write

$$\partial_t \rho = \{\rho, H\} = \int d\mathbf{x}'\left[\{\rho, \gamma_k'\}\frac{\delta T}{\delta \gamma_k'} + \{\rho, \rho'\}\frac{\delta U}{\delta \rho'}\right]. \qquad \text{(A7)}$$

The comparison of Eq. (A7) with the continuity condition (A2) leads us to

$$\int d\mathbf{x}'\left[\{\rho, \gamma_k'\}\frac{\delta T}{\delta \gamma_k'} + \{\rho, \rho'\}\frac{\delta U}{\delta \rho'}\right] + v_k \partial_k \rho = 0. \qquad \text{(A8)}$$

We next introduce a local term in the integrand by using δ-function, and express the velocity components v_l in terms of the functional derivatives $\delta T / \delta \gamma_k$ as

$$v_l = \frac{\delta T}{\delta \pi_l} = \int d\mathbf{x}' \frac{\delta T}{\delta \gamma_k'}\frac{\delta \gamma_k'}{\delta \pi_l} = e^{lki}\partial_k \frac{\delta T}{\delta \gamma_i}, \qquad \text{(A9)}$$

which can be directly obtained from Eq. (A4). Upon integrating by parts and after some algebra in Eq. (A8), we obtain

$$\int d\mathbf{x}' \frac{\delta T}{\delta \gamma_k'}[\{\rho, \gamma_k'\} - e^{kml}\partial_l \rho \partial_m \delta(\mathbf{x} - \mathbf{x}')] + \int d\mathbf{x}' \{\rho, \rho'\}\frac{\delta U}{\delta \rho'} = 0.$$

This implies that

$$\{\rho, \gamma_k'\} = e^{kml}\partial_l \rho \partial_m \delta(\mathbf{x} - \mathbf{x}'), \quad \{\rho, \rho'\} = 0. \qquad \text{(A10)}$$

Now we need to compute the Poisson bracket $\{\gamma_i,\gamma_k'\}$. By using the same reasoning as with the density, we can write the equation of motion for the vorticity of momentum density, $\boldsymbol{\gamma}$, as

$$\partial_t\gamma_i=\{\gamma_i,H\}=\int d\mathbf{x}'\left[\{\gamma_i,\gamma_k'\}\frac{\delta T}{\delta\gamma_k'}+\{\gamma_i,\rho'\}\frac{\delta T}{\delta\rho'}\right]+\{\gamma_i,U\}. \tag{A11}$$

With bracket $\{\rho,\gamma_k'\}$ already computed and

$$\frac{\delta T}{\delta\rho}=\frac{1}{2}v_k^2,$$

Eq. (A11) can be rewritten as

$$\partial_t\gamma_i=\int d\mathbf{x}'\{\gamma_i,\gamma_k'\}\frac{\delta T}{\delta\gamma_k'}-e^{iml}\partial_m\left(\frac{1}{2}v_k^2\partial_l\rho\right)+\{\gamma_i,U\}. \tag{A12}$$

By comparing Eqs. (A12) with Eq. (A6), we obtain

$$\int d\mathbf{x}'\{\gamma_i,\gamma_k'\}\frac{\delta T}{\delta\gamma_k'}-e^{imn}\partial_m(e^{nkl}v_k\gamma_l)+\{\gamma_i,U\}-e^{imn}\partial_m f_n=0.$$

If we introduce the local term $e^{imn}\partial_m(e^{nkl}v_k\gamma_l)$ into the integral using δ-function and replace velocity components v_l in accordance with Eq. (A9), we obtain, after integration by parts, that

$$\int d\mathbf{x}'\frac{\delta T}{\delta\gamma_k'}[\{\gamma_i,\gamma_k'\}-e^{ipj}e^{jln}e^{kmn}\partial_p\gamma_l\partial_m\delta(\mathbf{x}-\mathbf{x}')]+\{\gamma_i,U\}$$
$$-e^{imn}\partial_m f_n=0.$$

This immediately implies that the Poisson bracket for vector field $\boldsymbol{\gamma}$ and the relation between the exterior force and the potential energy are given by

$$\{\gamma_i,\gamma_k'\}=e^{ipj}e^{jln}e^{kmn}\partial_p\gamma_l\partial_m\delta, \tag{A13}$$

$$\{\gamma_i,U\}=e^{imn}\partial_m f_n. \tag{A14}$$

We note that the resulting force $\mathbf{f}$ can be found from Eq. (A14) up to a gradient term. This fact is a consequence of the invariance of the equations of motion (A1)–(A3) under the transformation $p\to p+\phi$, $f_i\to f_i-\partial_i\phi$, where ϕ is an arbitrary function whose choice has no influence on physical implications of the theory. Thus it follows from Eq. (A14) that no structure other than $f_i=\partial_i(\delta U/\delta\rho)$ is admissible to serve as an external force in the case where $U=U[\rho]$.

By collecting Eqs. (A10) and (A13), we find the complete system of Poisson brackets in the phase space (γ,ρ):

$$\{\rho,\rho'\}=0, \tag{A15}$$

$$\{\rho,\gamma_k'\}=e^{kml}\partial_l\rho\partial_m\delta, \tag{A16}$$

$$\{\gamma_i,\gamma_k'\}=e^{ipj}e^{jln}e^{kmn}\partial_p\gamma_l\partial_m\delta. \tag{A17}$$

Therefore, the equations of motion for an incompressible nonuniform fluid corresponding to these Poisson brackets, take form

$$\partial_t\gamma=\{\gamma,H\}=\nabla\times\left(\left[\gamma,\nabla\times\frac{\delta H}{\delta\gamma}\right]+\frac{\delta H}{\delta\rho}\nabla\rho\right), \tag{A18}$$

$$\partial_t\rho=\{\rho,H\}=-\left(\nabla\times\frac{\delta H}{\delta\gamma}\right)\nabla\rho. \tag{A19}$$

Equations (A15)–(A19) may be used as the fundamental principle in constructing a hierarchy of reduced Poisson brackets for various models of contour dynamics.

APPENDIX B: HAMILTONIAN VERSION OF CONTOUR DYNAMICS IN 2D PLANE

We consider a two-dimensional plane flow where the curl of the momentum is normal to the flow plane and hence has the only component:

$$\gamma=\{0,0,\gamma\},\quad \gamma=\varepsilon^{ik}\partial_i\pi_k, \tag{B1}$$

where ε^{ik} is the unit antisymmetric tensor of the second rank (with $\varepsilon^{12}=1$). In this case the Poisson brackets (A15)–(A17) for an incompressible inhomogeneous fluid can be reformulated for the dynamical variables γ, ρ as

$$\{\rho,\rho'\}=0, \tag{B2}$$

$$\{\rho,\gamma'\}=\varepsilon^{ki}\partial_i\rho\partial_k\delta(\mathbf{x}-\mathbf{x}'), \tag{B3}$$

$$\{\gamma,\gamma'\}=\varepsilon^{ki}\partial_i\gamma\partial_k\delta(\mathbf{x}-\mathbf{x}'). \tag{B4}$$

It is well known that two-dimensional dynamics of patches of constant vorticity and density can be reduced to dynamics of their contours, while ignoring the description of the rest of the fluid. However, it is a nontrivial fact that the description of the contour evolution can take various forms depending on the variables used; this deserves attention from both practical and theoretical standpoints.

For simplicity, we consider a single domain G^+ bounded by a closed fluid contour that separates it from the rest of the fluid in an exterior region G^-. By denoting the vorticity and the density inside and outside the contour as ω^+, ρ^+, and ω^-, ρ^-, we use the respective + and − superscripts to label variables in the internal domain G^+ and in the exterior region G^-. Using this notation, momentum and mass density can be written as

$$\pi=\rho^+\mathbf{v}^+\theta^++\rho^-\mathbf{v}^-\theta^-,\quad \rho=\rho^+\theta^++\rho^-\theta^-. \tag{B5}$$

where θ^+ and θ^- are mutually-complementary substantive functions defined as

$$\theta^+=\begin{cases}1, & \text{if } \mathbf{x}\in G^+,\\ 0, & \text{if } \mathbf{x}\in G^-,\end{cases}\qquad \theta^-=\begin{cases}1, & \text{if } \mathbf{x}\in G^-,\\ 0, & \text{if } \mathbf{x}\in G^+,\end{cases}$$

such that

$$\theta^++\theta^-=1,\quad \theta^+\theta^-=0. \tag{B6}$$

We note that a substantive θ-function characterizing a fluid domain, by definition, has dynamical property:

$$\partial_t\theta+v_k\partial_k\theta=0,$$

which implies that the corresponding domain moves together with the fluid.

Inserting π-representation (B5) into Eq. (B1) yields

$$\gamma = \rho^+\omega^+\theta^+ - \rho^-\omega^-\theta^- + \beta, \tag{B7}$$

where variable β can be expressed as

$$\beta = (\rho^+ v_k^+ - \rho^- v_k^-)\varepsilon^{ik}\partial_i\theta^+. \tag{B8}$$

It is easily seen that β has δ-functional character and thus describes a vortex sheet whose density is specified by the jump of the tangential momentum across the contour.

As the first step, we transform Poisson brackets (B2)–(B4) from the phase space (γ,ρ) into the space of dynamical variables (β,θ^+). In accordance with Eqs. (B5)–(B7), we have

$$\rho = \rho^- + (\rho^+ - \rho^-)\theta^+, \tag{B9}$$

$$\gamma = \rho^-\omega^- + (\rho^+\omega^+ - \rho^-\omega^-)\theta^+ + \beta. \tag{B10}$$

Depending on the existence (or the absence) of the mass density jump across the contour, substitution of Eqs. (B9) and (B10) into Eqs. (B2)–(B4) leads to two possible types of Poisson brackets. The detailed consideration of piecewise-constant plane vortex models without $(\rho^+=\rho^-)$ and with $(\rho^+ \neq \rho^-)$ jumps of mass density is provided in works [15,16].

APPENDIX C: COVARIANT FORMULATION

Description of the contour evolution can be complicated by a poor choice of generalized coordinates. When hydrodynamical motions possess some spatial symmetry, it is rational to use curvilinear coordinates corresponding to this symmetry. Therefore, in the context of our paper, we use as an example cylindrical coordinates x,r,ϕ, and consider the Hamiltonian formulation that is independent of the choice of the coordinate frame of reference. The transformation of hydrodynamic equations of motion are transformed from one of coordinate system into the other is accomplished in the framework of Poisson brackets transformations.

Let us now give a covariant formulation of expressions

$$\{\rho,\rho'\} = 0, \tag{C1}$$

$$\{\rho,\gamma_k'\} = e^{kml}\partial_l\rho\partial_m\delta(\mathbf{x}-\mathbf{x}'), \tag{C2}$$

$$\{\gamma_i,\gamma_k'\} = e^{ipj}e^{jln}e^{kmn}\partial_p\gamma_l\partial_m\delta(\mathbf{x}-\mathbf{x}'), \tag{C3}$$

which has been previously written in the Cartesian coordinates.

We introduce curvilinear coordinates $\boldsymbol{\zeta}=(\zeta^1,\zeta^2,\zeta^3)$ considered to be functions of Cartesian coordinates, supposing that a one-to-one transformation exists

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\zeta}) \leftrightarrows \boldsymbol{\zeta} = \boldsymbol{\zeta}(\mathbf{x}). \tag{C4}$$

It is well-known that covariant and contravariant metric tensors are defined by expressions

$$g_{ik} = \frac{\partial x^l}{\partial\zeta^i}\frac{\partial x^l}{\partial\zeta^k}, \leftrightarrows g^{ik} = \frac{\partial\zeta^i}{\partial x^l}\frac{\partial\zeta^k}{\partial x^l}.$$

The rules of scalar and vector quantity modifications under such transformations (C4) are as follows:

$$f(\boldsymbol{\zeta}) = \int d\mathbf{x}\, f(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}(\boldsymbol{\zeta})), \tag{C5}$$

$$u^k(\boldsymbol{\zeta}) = \int d\mathbf{x}\frac{\partial\zeta^k}{\partial x^i}u^i(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}(\boldsymbol{\zeta})), \tag{C6}$$

$$u_k(\boldsymbol{\zeta}) = g_{km}u^m(\boldsymbol{\zeta}) = \int d\mathbf{x}\frac{\partial x^i}{\partial\xi^k}u^i(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}(\boldsymbol{\zeta})). \tag{C7}$$

To be practical, these rules are completed by the basic identity

$$e^{iml}\frac{\partial\zeta^k}{\partial x^i}\frac{\partial\zeta^p}{\partial x^m}\frac{\partial\zeta^n}{\partial x^l} \equiv g^{-1/2}e^{kpn}, \tag{C8}$$

from which it follows that

$$\frac{\partial\zeta^k}{\partial x^i}e^{iml} \equiv g^{-1/2}e^{kpn}\frac{\partial x^m}{\partial\zeta^p}\frac{\partial x^l}{\partial\zeta^n}, \tag{C9}$$

$$e^{iml}\frac{\partial\zeta^k}{\partial x^i}\frac{\partial\zeta^p}{\partial x^m} \equiv g^{-1/2}e^{kpn}\frac{\partial x^l}{\partial\zeta^n}, \tag{C10}$$

where $e^{ikj} \equiv e_{ikj}$ is the completely anti-symmetrical tensor, known as Levi-Civitta tensor, ($e^{ikj}=e_{ikj}=1$, when $ikj=123,231,312$; $e^{ikj}=e_{ikj}=-1$, when $ikj=321,132,213$; and $e^{ikj}=e_{ikj}=0$, when any two indices are equal). Keeping in mind Eq. (C6), let us note that identity (C8) is the transformation of tensor e^{ikj} upon transformation (C4).

In particular, Eqs. (C5) and (C6) give the following expressions for density ρ and vorticity component γ^j of a hydrodynamic momentum in curvilinear coordinates:

$$\rho(\boldsymbol{\zeta}) = \int d\mathbf{x}\,\rho(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}(\boldsymbol{\zeta})), \tag{C11}$$

$$\gamma^k(\boldsymbol{\zeta}) = \int d\mathbf{x}\frac{\partial\zeta^k}{\partial x^i}\gamma^j(\mathbf{x})\,\delta(\mathbf{x}-\mathbf{x}(\boldsymbol{\zeta})). \tag{C12}$$

Using the definition for $\boldsymbol{\gamma}=\mathrm{rot}\,\boldsymbol{\pi}$ in Cartesian coordinates, we can now easily find from Eq. (C12) that the covariant generalization of this expression in curvilinear coordinates is

$$\gamma^k(\boldsymbol{\zeta}) = \int d\mathbf{x}\frac{\partial\zeta^k}{\partial x^i}e^{iml}\frac{\partial\pi_l}{\partial x^m}\delta(\mathbf{x}-\mathbf{x}(\boldsymbol{\zeta})) = g^{-1/2}e^{kpn}\frac{\partial\pi_n}{\partial\zeta^p}. \tag{C13}$$

Here, g is determinant of the metric tensor, and $\pi_n(\boldsymbol{\zeta}) = \rho u_n(\boldsymbol{\zeta})$ are covariant components of the hydrodynamical moment in curvilinear coordinates $\boldsymbol{\zeta}$.

A covariant generalization for the Poisson brackets (C1)–(C3) may be found, using Eqs. (C11) and (C12), from the following commutative relations:

$$\{\rho(\boldsymbol{\zeta}),\rho(\boldsymbol{\zeta}')\} = \int d\mathbf{x}\,d\mathbf{x}'\{\rho(\mathbf{x}),\rho(\mathbf{x}')\}\delta(\mathbf{x}-\mathbf{x}(\boldsymbol{\zeta}))\,\delta(\mathbf{x}'-\mathbf{x}(\boldsymbol{\zeta}')), \tag{C14}$$

$$\{\rho(\boldsymbol{\zeta}),\gamma^k(\boldsymbol{\zeta}')\} = \int d\mathbf{x}\, d\mathbf{x}' \frac{\partial \zeta'^k}{\partial x'^m}\{\rho(\mathbf{x}),\gamma^m(\mathbf{x}')\}\delta(\mathbf{x}-\mathbf{x}(\boldsymbol{\zeta})) \times \delta(\mathbf{x}'-\mathbf{x}(\boldsymbol{\zeta}')), \quad \text{(C15)}$$

$$\{\gamma^j(\boldsymbol{\zeta}),\gamma^k(\boldsymbol{\zeta}')\} = \int d\mathbf{x}\, d\mathbf{x}' \frac{\partial \zeta^i}{\partial x^n}\frac{\partial \zeta'^k}{\partial x'^m}\{\gamma^n(\mathbf{x}),\gamma^m(\mathbf{x}')\} \times \delta(\mathbf{x}-\mathbf{x}(\boldsymbol{\zeta}))\,\delta(\mathbf{x}'-\mathbf{x}(\boldsymbol{\zeta}')). \quad \text{(C16)}$$

By putting Eqs. (C1)–(C3) for brackets (which are under the integral signs), and by using the formulas

$$\frac{\partial \zeta^i}{\partial x^n} e^{npj}\frac{\partial}{\partial x^p} = g^{-1/2} e^{ipn}\frac{\partial x^j}{\partial \zeta^n}\frac{\partial}{\partial \zeta^p},$$

$$e_{jtl}\frac{\partial x^j}{\partial \zeta^n}\frac{\partial x^t}{\partial \zeta^s} = g^{1/2} e_{nsj}\frac{\partial \zeta^j}{\partial x^l},$$

which follow immediately from Eq. (C9) and (C10), after some transformations we obtain that

$$\{\gamma^j(\boldsymbol{\zeta}),\gamma^k(\boldsymbol{\zeta}')\} = \int d\mathbf{x}\, d\mathbf{x}' \frac{\partial \zeta^i}{\partial x^n}\frac{\partial \zeta'^k}{\partial x'^m} e^{npj} e_{jlt} e^{mts} \times \frac{\partial}{\partial x^p}\left(\gamma^l \frac{\partial}{\partial x^s}\delta(\mathbf{x}-\mathbf{x}')\right)\delta(\mathbf{x}-\mathbf{x}(\boldsymbol{\zeta})) \times \delta(\mathbf{x}'-\mathbf{x}(\boldsymbol{\zeta}')) = g^{-1/2} e^{ipn} e_{jsn} e^{ksm}\frac{\partial}{\partial \zeta^p}\left(\gamma_j \frac{\partial}{\partial \zeta^m} g^{1/2}\delta(\zeta-\boldsymbol{\zeta}')\right). \quad \text{(C17)}$$

Ultimately, the Poisson brackets for contravariant components γ^k and density ρ are found from the brackets (C1)–(C3) as a result of the coordinate transformation (C4)

$$\{\rho(\boldsymbol{\zeta}),\rho(\boldsymbol{\zeta}')\} = 0, \quad \text{(C18)}$$

$$\{\rho(\boldsymbol{\zeta}),\gamma^k(\boldsymbol{\zeta}')\} = g^{-1/2} e^{knm}\frac{\partial}{\partial \zeta^m}\rho\frac{\partial}{\partial \zeta^n} g^{-1/2}\delta(\boldsymbol{\zeta}-\boldsymbol{\zeta}'), \quad \text{(C19)}$$

$$\{\gamma^j(\boldsymbol{\zeta}),\gamma^k(\boldsymbol{\zeta}')\} = g^{-1/2} e^{ipj} e_{jln} e^{knm}\frac{\partial}{\partial \zeta^p}\gamma^l\frac{\partial}{\partial \zeta^m} g^{-1/2}\delta(\boldsymbol{\zeta}-\boldsymbol{\zeta}'). \quad \text{(C20)}$$

The derived Poisson brackets (C1)–(C3) and their covariant analog (C18)–(C20) can be used in various versions of the Hamiltonian description of the contour dynamics models.

APPENDIX D: HAMILTONIAN FOR NONPLANAR MODELS OF CONTOUR DYNAMICS

For the models of incompressible inhomogeneous fluid, the Hamiltonian is usually chosen a priori as the kinetic energy of the flow:

$$H = \frac{1}{2}\int d\boldsymbol{\zeta} \quad g^{1/2}\rho v^i v_i. \quad \text{(D1)}$$

In one Hamiltonian version of contour dynamics, it is reasonable to express this integral in terms of dynamic variables which are defined on the contour.

With this purpose, we use the incompressibility condition as the first step

$$\frac{\partial}{\partial \zeta^i}(g^{1/2} v^i) = 0. \quad \text{(D2)}$$

The stream function, $\boldsymbol{\psi}$ (a vector variable, i.e., a vector potential), which is connected with covariant components of velocity, is introduced by the following relationship

$$v^i = -g^{-1/2} e^{ikn}\frac{\partial \psi_n}{\partial \zeta^k}. \quad \text{(D3)}$$

If we substitute Eq. (D2) into Eq. (D1) and integrate in parts, we find that the Hamiltonian, H, can be expressed in terms of vorticity $\boldsymbol{\gamma}=\text{rot}\,\rho\mathbf{v}$of the hydrodynamical moment and stream function $\boldsymbol{\psi}$ as

$$H = -\frac{1}{2}\int d\zeta\, g^{1/2}\gamma^j\psi_i.$$

We assume now that in the chosen curvilinear system of coordinates, vector fields $\boldsymbol{\gamma}$ and $\boldsymbol{\psi}$ have nonzero only the third components, $\gamma^3=\gamma$ and $\psi_3=\psi$. In this case, we come to the expression

$$H = -\frac{1}{2}\int d\zeta_1 d\zeta_2 g^{1/2}\gamma\psi. \quad \text{(D4)}$$

Here, ζ_1 and ζ_2 are the corresponding curvilinear coordinates, and ψ is expressed via γ using Eq. (8).

[1] In their works, E. Fermi and J. von Neumann used the term "Taylor instability" for this type of instability [see Refs. [3–5]: "Excerpt from a Lecture on Taylor instability, Given during the Fall of 1951 at Los Alamos Scientific Laboratory" (paper [3]); "Taylor Instability of an Incompressible Liquid" (paper [4]); "Taylor Instability at the Boundary of Two Incompressible Liquids" (paper [5])]. In the modern literature, some researchers use the term "Rayleigh-Taylor instability." This is how Fermi describes the process (paper [3]): "Taylor instability is the instability of the surface between two fluids when the heavier material is being accelerated by the lighter material. It is traditionally illustrated by the instability of the surface between air and water when a beaker of water is turned upside down. The adequacy of this example can be seen by supposing the beaker to be in an elevator in a gravitationless region. If the elevator were accelerated in a direction corresponding to a

vector drawn from the closed end to the open end of the beaker the surface would be stable. This acceleration corresponds to a gravitational field in the opposite direction (towards the closed end or bottom of the beaker). If the elevator were accelerated in a direction corresponding to a vector drawn from the surface of the liquid to the closed end of the beaker the surface would be unstable. This case corresponds to a gravitational field towards the open end of the beaker; i.e., when the beaker is turned upside down."
[2] E. Fermi, in *Scientific Works of E. Fermi*, edited by B. Pontecorvo and V. Pokrovskii (Nauka, Moscow, 1972), p. 588 [in Russian]; see also *Collected Papers of Enrico Fermi* (University of Chicago Press, Chicago, 1965).
[3] E. Fermi, *Collected Papers of Enrico Fermi* (University of Chicago Press, Chicago, 1965), Vol. II, Chap. 243, p. 813.
[4] E. Fermi, *Collected Papers of Enrico Fermi* (Ref. [3]), Vol. II, Chap. 244, p. 816.
[5] E. Fermi and J. von Neumann, *Collected Papers of Enrico Fermi*, (Ref. [3]), Vol. II, Chap. 245, p. 821.
[6] *Collected Papers of Enrico Fermi* (University of Chicago Press, Chicago, 1965), comments by S. Chandrasekhar to Chaps. 261, 262, p. 925. In view of the overall complexity of hydrodynamical problems, the great Enrico Fermi insightfully remarked that it is so easy to make a mistake in this domain that one should not believe in a result obtained after a long and complicated mathematical derivation if one cannot understand its physical origin; in the same way, one cannot also believe in a long and complicated piece of physical reasoning if one cannot demonstrate them mathematically.
[7] N. A. Inogamov, A. Yu. Demyanov, and E. E. Son, *Hydrodynamics of Mixing* (Mosk. Fiz.-Tekh. Inst., Moscow, 1999) [in Russian].
[8] M. Lavrentiev and B. Chabat, *Effets Hydrodynamiques et Modèles Mathématiques* (Mir, Moscow, 1980).
[9] N. J. Zabusky, M. H. Hughes, and K. V. Roberts, J. Comput. Phys. **30**, 96 (1979).
[10] D. G. Dritschel, J. Comput. Phys. **79**, 477 (1988).
[11] D. G. Dritschel and M. E. McIntyre, Phys. Fluids A **2**, 748 (1990).
[12] D. I. Pullin, J. Fluid Mech. **108**, 401421 (1981).
[13] D. I. Pullin, Annu. Rev. Fluid Mech. **24**, 84115 (1991).
[14] P. G. Saffman, *Vortex Dynamics* (Cambridge University Press, Cambridge, UK, 1992, 1995; Nauchny Mir, Moscow, 2000).
[15] V. P. Goncharov and V. I. Pavlov, Zh. Eksp. Teor. Fiz. **119**, 685 (2001) [JETP **92**, 594 (2001)].
[16] V. P. Goncharov and V. I. Pavlov, in *Fundamental and Applied Problems of the Vortex Theory*, edited by A. Borisov, I. Mamaev, and M. Sokolovskii (Inst. Kompyut. Issled., Moscow Izhevsk, 2003) [in Russian].
[17] V. P. Goncharov, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza **2**, 65 (1986).
[18] V. P. Goncharov and V. I. Pavlov, Eur. J. Mech. B/Fluids **19**, 831 (2000).
[19] V. P. Maslov, *Operator Methods* (Mir, Moscow 1976) [translated from Russian].
[20] Yu. A. Dubinskii, Russ. Math. Surveys **37**, 109 (1982).
[21] To avoid misunderstandings, let us remind that covariant components of vorticity, ω_j (and its respective contravariant ω^j) are defined by the relation $\omega=\omega_j\mathbf{e}^j(=\hat{\omega}^j\mathbf{e}_j)$ on the basis $\mathbf{e}^j$ (or $\mathbf{e}_j$) using standard definitions via metrical and Kronecker's tensors, $\mathbf{e}_i\cdot\mathbf{e}_j=g_{ij},\mathbf{e}^i\cdot\mathbf{e}^j=g^{ij}$, and $\mathbf{e}_i\cdot\mathbf{e}^j=\delta_i^j$. If in the framework of the chosen cylindrical coordinate system, x,r,φ, only the azimuthal component is nonzero, then $\omega_i=(0,\Omega,0)$, where $\Omega=r^{-1}(-\partial_2 v_1+\partial_1 v_2)$ and the contravariant component is $\hat{\Omega}=r(-\partial_2 v_1+\partial_1 v_2)$. The "physical" component has the same dimension as the vorticity vector, ω, and equals $\omega=(-\partial_2 v_1+\partial_1 v_2)$. The dimensions of these three components are different: $[\omega]=T^{-1}$, $[\Omega]=L^{-1}T^{-1}$, and $[\hat{\Omega}]=LT^{-1}$.
[22] L. M. Milne-Thomson, *Theoretical Hydrodynamics*, Vth ed. (MacMillan, London, 1968).
[23] Y. A. Li and P. J. Olver, Discrete Contin. Dyn. Syst. **3**, 419 (1997).
[24] P. Rosenau, Not. Am. Math. Soc. **55**, 738 (2005).
[25] V. E. Zakharov and E. A. Kuznetsov, Zh. Eksp. Teor. Fiz. **91**, 1310 (1986) [Sov. Phys. JETP **64**, 773 (1986)].
[26] E. A. Kuznetsov, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **56**, 342 (2003).
[27] V. Goncharov and V. Pavlov, Eur. J. Mech. B/Fluids **16**, 509 (1997).
[28] The formalism of noncanonical Poisson brackets began developing in the late 1970s, and was applied to the construction and description of hydrodynamic field systems. See reviews of I. Dzjaloshinski and G. Volovik [Ann. Phys. (N.Y.) **125**, 67 (1980)], R. Littlejohn [in *Mathematical Methods in Hydrodynamics and Integrability in Related Dynamical Systems*, edited by M. Tabor and Y. M. Treve, AIP Conf. Proc. No. 88 (AIP, New York, 1982), p. 47], P. Morrison [in *Mathematical Methods in Hydrodynamics and Integrability in Related Dynamical Systems*, edited by, M. Tabor and Y. M. Treve, AIP Conf. Proc. No. 88 (AIP, New York, 1982), p. 13], S. Novikov [Russ. Math. Surveys **37**, 1 (1982)].
[29] The gradient transformation in Eq. (60) is a trivial case of the more general transformation $z\rightarrow z+\hat{L}(q)$, where $\hat{L}$ is any operator possessing property $\int ds\, f\hat{L}f=0$.
[30] This integral encompasses the fluid captured by compacton and is conserved due to the incompressibility condition.
[31] Here we use the well-known fact that Lagrangian is determined up to a full temporal derivative of any function of coordinates and time, and that multiplication of a Lagrangian on an arbitrary constant does not modify the equations of motion.